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A Note on Type 2 Generalized Laplacian Family

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ABSTRACT
In a recent paper Sebastian and Gorenflo (2016) introduced Type 2 Generalized Laplacian (T2GL) law and developed the associated AR(1) model after showing that T2GL law belongs to class-L. Here we show that T2GL law is normally attracted to a stable law and it is geometrically infinitely divisible. In fact we prove the results for a T2GL family that contains the T2GL law. We point out the corresponding integer-valued T2GL family. Finally, we also clarify a claim in Sebastian and Gorenflo (2016).

KEYWORDS
AR(1) model, α-decomposable distributions, characteristic function, class-L, discrete stable laws, geometric infinite divisibility, Laplace transform, normal attraction, probability generating function, stable laws

1. Introduction

The T2GL(α) random variable ($r.v.$) X is described as $X = X_1 - X_2$ where X_1 and X_2 are independent Mittag-Leffler(α) (ML(α)) $r.v.$s with Laplace Transform (LT)

$$\frac{1}{1+s^{\alpha}},\ \alpha \in (0,1].$$

Historically, the ML distribution, following Pillai (1990), was first described by Kovalenko (1965). The characteristic function (CF) of T2GL(α) law is

$$\frac{1}{1+2\ \cos(\frac{\pi\alpha}{2})\ |t|^{\alpha}+|t|^{2\alpha}},\ \alpha \in (0,1].$$

Sebastian and Gorenflo (2016), who introduced T2GL law, prove that it belongs to class-L (or is self-decomposable). They then develop the corresponding first order

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auto-regressive (AR(1)) model and study it in some detail.

In the next section we show that T2GL law belongs to the domain of normal attraction of a stable law and is geometrically infinitely divisible (GID). Actually we prove these properties for a family of laws that contains T2GL. We point out the corresponding family of integer-valued T2GL laws that share these properties. Finally, we clarify a claim in Sebastian and Gorenflo (2016).

Several authors have discussed applications of GID laws in time series modelling, p-thinning of renewal processes and random walks, see Sandhya *et al.* (2018) and the references therein. Random time changed Lévy (stationary and independent increment) processes are extensively used to model finance data exhibiting stochastic volatility, see Schoutens (2003). GID laws are the increments of exponential/ gamma time changed Lévy processes. Here we have another distribution for the random time change. Thus the results in this note will play a useful role in stochastic modelling of data consisting of both positive, negative and discrete values.

2. Divisibility properties of T2GL laws

Introducing a scale parameter we have the LT of the ML(α, λ) as

$$\frac{1}{1+\lambda s^{\alpha}},\ \alpha \in (0,1],\ \lambda > 0.$$

The corresponding T2GL(α, λ) laws have CF

$$\frac{1}{1+2\ \cos(\frac{\pi\alpha}{2})\ \lambda|t|^{\alpha}+\lambda^2|t|^{2\alpha}},\ \alpha \in (0,1],\ \lambda > 0.$$

It is known (Pillai (1990)) that ML(α) law is normally attracted to stable(α) laws. Let us see whether T2GL law has any such property. Let $S_n = X_1 + \cdots + X_n$ where $X_i; i = 1 \cdots, n$ are independent and identically distributed (*i.i.d.*) T2GL(α, λ) *r.v.*s. Then the CF of $n^{-1/\alpha} S_n$ is

$$\left\{1+\frac{1}{n}\left[2\lambda\cos\left(\frac{\pi\alpha}{2}\right)|t|^{\alpha}+\frac{\lambda^2}{n}|t|^{2\alpha}\right]\right\}^{-n}.$$

We know that as $n \to \infty$, $(1+\frac{1}{n}x)^{-n} \to \exp(-x)$ uniformly in x and here $2\lambda\cos\left(\frac{\pi\alpha}{2}\right)|t|^{\alpha}+\frac{\lambda^2}{n}|t|^{2\alpha} \to 2\lambda\cos\left(\frac{\pi\alpha}{2}\right)|t|^{\alpha}$. Hence

$$\left\{1+\frac{1}{n}\left[2\lambda\cos\left(\frac{\pi\alpha}{2}\right)|t|^{\alpha}+\frac{\lambda^2}{n}|t|^{2\alpha}\right]\right\}^{-n} \to e^{-\left[2\lambda\cos\left(\frac{\pi\alpha}{2}\right)|t|^{\alpha}\right]}$$

proving

Proposition 2.1. *T2GL(α, λ) law is normally attracted to stable(α) law.*

Remark 1. This also follows from the facts (i) that ML(α) laws are normally attracted to stable(α) laws and (ii) that if X_1 and X_2 are independent *r.v.*s normally attracted to stable(α) so is $X_1 - X_2$.

That ML laws are GID is used in developing certain AR(1) and p-thinning models, see Sandhya, *et al.* (2018) for a review on this. Random time changed Lévy processes are used to model finance data exhibiting stochastic volatility, see Schoutens (2003). Since GID laws are the increments of exponential/ gamma time changed Lévy processes, GID laws can also be used to model exponential/ gamma volatility. So one may be interested to know whether T2GL laws are GID. This is addressed in Proposition 2.6. In this attempt we also get another distribution for the random time change. A CF ϕ is GID if for every $p \in (0,1)$ there is a CF ξ_p such that

$$\phi(t) = \frac{p\xi_p(t)}{1-(1-p)\xi_p(t)} \implies \xi_p(t) = \frac{\phi(t)}{p+q\phi(t)},\ q = 1-p.$$

Equivalently, it is enough to check whether or not $\frac{\phi(t)}{p+q\phi(t)}$ is a CF for every $p \in (0,1)$. Taking $\phi(t) = \left(1 + 2\cos\left(\frac{\pi\alpha}{2}\right)\lambda|t|^\alpha + \lambda^2|t|^{2\alpha}\right)^{-1},\ \alpha \in (0,1], \lambda > 0$, we get,

$$\xi_p(t) = \frac{1}{1 + 2\cos\left(\frac{\pi\alpha}{2}\right)p\lambda|t|^\alpha + p\lambda^2|t|^{2\alpha}}.$$

If ξ_p is a CF for every $p \in (0,1)$, then ϕ is GID. To check whether ξ_p is a CF we proceed as follows. Consider the function $\psi(s) = \lambda_1 s^{\alpha_1} + \lambda_2 s^{\alpha_2},\ s \geq 0,\ \lambda_i > 0,\ 0 < \alpha_i \leq 1,\ i = 1,2$. This function ψ is non-negative, $\psi(0) = 0$ and has completely monotone (CM) derivative, since both the terms have CM derivatives. This is because when we differentiate them the terms in the sum ψ change their signs simultaneously, guaranteeing that ψ has CM derivative. Since $1/(1+s)$ is CM, by the criterion 2 in Feller, 1971, p.441, we get that

$$\frac{1}{1+\lambda_1 s^{\alpha_1} + \lambda_2 s^{\alpha_2}},\ s \geq 0,\ \lambda_i > 0,\ 0 < \alpha_i \leq 1,\ i = 1,2,$$

is CM. Since this function when evaluated at $s = 0$ equals one, we have

Lemma 2.2. $1/(1 + \lambda_1 s^{\alpha_1} + \lambda_2 s^{\alpha_2}),\ \lambda_i > 0,\ 0 < \alpha_i \leq 1,\ i = 1,2,$ *is the LT of some probability law.*

Since e^{-s} is CM we get, by the same criterion 2 in Feller, 1971, p.441, the following result which also follows as it is the LT of the sum of two independent stable laws.

Lemma 2.3. $e^{-(\lambda_1 s^{\alpha_1} + \lambda_2 s^{\alpha_2})},\ \lambda_i > 0,\ 0 < \alpha_i \leq 1,\ i = 1,2,$ *is the LT of some probability law.*

We first prove a general result and derive the CFs corresponding to the LTs in Lemmas 2.2 and 2.3 from it. Consider a symmetric stable(α) *r.v.* $X(\lambda)$ with scaling parameter $\lambda^{1/\alpha},\ \lambda > 0,\ 0 < \alpha \leq 2$ and let λ be a *r.v.* with LT ϕ. Then the CF f of $X(\lambda)$ is

$$f(t) = E[e^{itX(\lambda)}] = E_\lambda[e^{-\lambda|t|^\alpha}] = \phi(|t|^\alpha),$$

and we have

Proposition 2.4. *If ϕ is a LT, then $f(t) = \phi(|t|^\alpha),\ 0 < \alpha \leq 2$ is a CF.*

Proposition 2.5. $f_1(t) = 1/(1 + \lambda_1|t|^{\beta_1} + \lambda_2|t|^{\beta_2}),\ \lambda_i > 0,\ 0 < \beta_i \leq 2,\ i = 1, 2,$ *is the CF of some probability law.*

We sketch a proof of this proposition that demonstrates a different possibility for randomization. Consider the symmetric stable(β) random variable $X(\lambda)$ with scaling parameter $\lambda^{1/\beta},\ \lambda > 0,\ 0 < \beta \leq 2$ and let λ be a *r.v.* with LT ϕ in Lemma 2.2. Then the CF f_1 of $X(\lambda)$ is

$$f_1(t) = E\left[e^{itX(\lambda)}\right] = E_\lambda\left[e^{-\lambda|t|^\beta}\right] = \phi(|t|^\beta) = \frac{1}{1 + \lambda_1|t|^{\alpha_1\beta} + \lambda_2|t|^{\alpha_2\beta}}.$$

Since $0 < \alpha_i \leq 1$ and $0 < \beta \leq 2$, we have $0 < \alpha_i\beta \leq 2$. Putting $\beta_i = \alpha_i\beta$ we get the CF f_1 in Proposition 2.5.

From Lemma 2.3 it follows on similar lines that $f_2(t) = e^{-(\lambda_1|t|^{\beta_1} + \lambda_2|t|^{\beta_2})},\ \lambda_i > 0,\ 0 < \beta_i \leq 2,\ i = 1, 2,$ is the CF of some probability law. This can also be seen as it is the product of two symmetric stable CFs. Since $(f_2)^u$ is a CF for every $u > 0$, f_2 is ID. Further, if we treat u to be a unit exponential *r.v*, then we get f_1. This is another way of proving Proposition 2.5.

Propositions 2.4 has interpretation in terms of randomized operational time in Lévy processes, see Feller (1971, p. 345, 451). For instance, the Lévy process with CF $\phi(\lambda_1|t|^{\alpha_1} + \lambda_2|t|^{\alpha_2})$ directed by the process with LT ϕ is subordinated to the Lévy process with CF f_2. In particular, the Lévy process with CF f_1 is subordinated to that with CF f_2 by the directing exponential process. On the other hand, Proposition 2.5 implies: the Lévy process with CF f_1 is subordinated to that with CF $\exp(-|t|^\beta), 0 < \beta \leq 2$ by the directing process with LT in Lemma 2.2. These interpretations are important as the resultant Lévy process can be derived from different Lévy processes using different subordinators, and we have another choice to model stochastic volatility in finance data.

From Proposition 2.5 it follows that

$$\xi_p(t) = \frac{1}{1 + 2\cos\left(\frac{\pi\alpha}{2}\right)p\lambda|t|^\alpha + p\lambda^2|t|^{2\alpha}}$$

is a CF for every $p \in (0, 1)$ and hence

Proposition 2.6. *$T2GL(\alpha, \lambda)$ law is GID.*

Calling the family of distributions with CF f_1 as $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ we can see that $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ law is also GID, which follows by a similar line of argument as above. Since every GID law is infinitely divisible (ID), see Sandhya (1990), Sandhya and Pillai (1999), we have

Proposition 2.7. *$T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ laws are GID, hence ID.*

Remark 2. Notice that the CF f_2 is not stable unless $\alpha_1 = \alpha_2$. Hence f_2 can be stable if and only if $\alpha_1 = \alpha_2 = \alpha$ in which case $f_2(t) = e^{-(\lambda_1+\lambda_2)|t|^{\alpha}},\ \lambda_i > 0,\ i = 1, 2,\ 0 < \alpha \le 2$, which is the CF of a symmetric stable(α) law. Consequently, f_1 above is not a geometrically stable CF unless $\alpha_1 = \alpha_2 = \alpha$ because of the one-to-one correspondence between stable laws and geometrically stable laws.

Having obtained the more general $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ family we check whether it is normally attracted to any stable law. The CF of $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ is $f_1(t) = \frac{1}{1+\lambda_1|t|^{\alpha_1}+\lambda_2|t|^{\alpha_2}}, \lambda_i > 0,\ 0 < \alpha_i \le 2,\ i = 1, 2\ \alpha_1 \ne \alpha_2$. Let $\alpha_2 > \alpha_1$. Then consider

$$\left[f_1(\frac{t}{n^{1/\alpha_1}})\right]^n = \left[\frac{1}{1 + (\frac{1}{n})\lambda_1|t|^{\alpha_1} + (\frac{1}{n^{\alpha_2/\alpha_1}})\lambda_2|t|^{\alpha_2}}\right]^n.$$

Since $\alpha_2 > \alpha_1$ the third term in the denominator goes to zero faster than $\frac{1}{n}$ so that

$$\left[f_1(\frac{t}{n^{1/\alpha_1}})\right]^n \to e^{-\lambda_1|t|^{\alpha_1}},\ as\ n \to \infty.$$

Note that if $\alpha_1 > \alpha_2$ similar steps lead to similar result. When $\alpha_1 = \alpha_2$ similar result obviously holds as already noted. Thus we have the following result that is stronger than Proposition 2.1.

Theorem 2.8. *$T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ law is normally attracted to stable(α) law where $\alpha = \min\{\alpha_1, \alpha_2\}$ with corresponding scale parameter.*

Remark 3. The $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ family has four parameters and thus is more flexible in modelling data than the T2GL law which has only one parameter.

2.1. Discrete Analogues

From Satheesh and Nair (2002) we have, if ϕ is a LT then $\phi((1 - s)),\ s \in (0, 1]$ is a probability generating function (PGF). From the LT in Lemma 2.2 we note that,

$$P(s) = 1/\left(1 + \lambda_1(1 - s)^{\alpha_1} + \lambda_2(1 - s)^{\alpha_2}\right),\ 0 < \alpha_i \le 1,\ \lambda_i > 0,\ i = 1, 2$$

is the PGF of discrete $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ law. This class of discrete laws share the divisibility properties (GID, ID) of its continuous counterpart. Here we briefly discuss the analogue of Theorem 2.8, since usually the notion of attraction is discussed only for continuous laws.

Steutel and van Harn (1979) conceived the notion of domain of attraction (DA) of discrete stable laws. Satheesh and Sandhya (2006) used a slightly different, equivalent description of this to discuss DA and domain of partial attraction of discrete laws. The PGF of discrete stable(α) law is $R(s) = \exp\{-\lambda(1 - s)^{\alpha}\}; 0 < \alpha \le 1,\ \lambda > 0$. For $\alpha = 1$ we have Poison(λ) law.

Definition 2.9. A discrete law with PGF P is in the DA of a discrete stable(α) law with PGF R if there exists constants $\{b_n \downarrow 0\}$ such that

$$\lim_{n\to\infty} \{P(1 - b_n s)\}^n = R(1 - s).$$

When $b_n = n^{-1/\alpha}$, we have normal attraction. Also, we have $P(1-b_n s) = 1/(1+\lambda_1(b_n s)^{\alpha_1} + \lambda_2(b_n s)^{\alpha_2}),\ \lambda_i > 0,\ 0 < \alpha_i \leq 1,\ i = 1,2$, when P is the PGF of discrete $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ law. With $b_n = n^{-1/\alpha},\ \alpha = \min\{\alpha_1, \alpha_2\}$ and proceeding along the lines leading to Theorem 2.8 we get

$$\lim_{n\to\infty}\{P(1-b_n s)\}^n = e^{-\lambda s^\alpha} = R(1-s) \implies R(s) = e^{-\lambda(1-s)^\alpha},$$

$\lambda > 0$ the corresponding scale parameter, proving the proposition given below.

Proposition 2.10. *Discrete $T2GL(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ law is normally attracted to discrete stable(α) law where $\alpha = \min\{\alpha_1, \alpha_2\}$ with corresponding scale parameter.*

3. Concluding Remarks

A sequence of *r.v.*s $\{X_n\}$ generates an AR(1) model with coefficient α, if $X_n = \alpha X_{n-1} + \epsilon_n,\ n \in \mathbf{Z}$, for some $\alpha \in (0,1)$ where ϵ_n is a sequence of *i.i.d.* *r.v.*s and ϵ_n is independent of X_n for each n.

A *r.v.* X is in class-L if $X \stackrel{D}{=} \alpha X + X_\alpha$, for every $\alpha \in (0,1)$ where X and X_α are independent and $\stackrel{D}{=}$ denotes equality in distribution.

Notice that the description of the AR(1) model needs the relation to be satisfied only "for some $\alpha \in (0,1)$" while for self-decomposability the relation is to be satisfied "for every $\alpha \in (0,1)$". Perhaps not noticing this lead Sebastian and Gorenflo (2016) to claim; "in Gaver and Lewis (1980) it is proved that only class-L distributions can be marginal distributions of a first order auto regressive process", while developing their AR(1) model associated with the T2GL model. This claim is incorrect for the following reasons; (i) Gaver and Lewis (1980) never made or proved such a statement (ii) the statement is wrong and (iii) the AR(1) model must be stationary even to have a connection between the two notions.

Gaver and Lewis (1980) were perhaps the first to notice the connection between class-L distributions and the marginals of stationary AR(1) models. We quote Gaver and Lewis (1980); "The limitation of the theory of class-L *r.v.*s as it relates to the present work (stationary AR(1) modelling) is that it requires a solution for each $0 < \alpha < 1,\ \cdots$. This full range of α is desirable, but may not occur." Thus they were quite clear in distinguishing the two notions.

A *r.v.* X is said to be α-decomposable if $X \stackrel{D}{=} \alpha X + X_\alpha$, for some $\alpha \in (0,1)$ where X and X_α are independent.

This notion of α-decomposable distributions dates back to Loève (1945). In Loève (1977) also this was discussed as part of Complements and Details: see 16 on Page 352 and there is notational confusion. His definition L_c as the family of all c-decomposable laws does not imply that L_1 is that of self-decomposable ones. The class of self-decomposable laws is $\bigcap_{0\leq c\leq 1} L_c$.

Clearly, the notion of α-decomposable laws is in tune with the structure of a stationary AR(1) model. Thus α-decomposable laws characterize the marginal distributions of a stationary AR(1) sequences with coefficient $\alpha \in (0, 1)$, see, Bouzar and Satheesh (2008), who further discussed the integer-valued analogue of α-decomposable laws and characterized the marginals of stationary INAR(1) models. They also gave a variety of examples to emphasize that the property of self-decomposability is not required to construct stationary AR(1) sequences, both in the continuous and in the discrete cases. The examples also stress that even finite range distributions qualify to be the marginals of stationary AR(1) sequences. See also, Satheesh and Sandhya (2007).

In this context it is worth noticing that whether the T2GL family is self-decomposable or at least α-decomposable for some α, is not yet ascertained.

One may consider generalized T2GL laws with LT $1/(1 + \sum_{i=1}^{k} \lambda_i s^{\alpha_i});\ 0 < \alpha_i \leq 2,\ \lambda_i > 0$ and derive properties similar to the above.

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